Introduction

Building model of reality and of dependencies in real problems we meet with the data that are uncertain, that we cannot determine in an accurate manner. In such cases, the uncertain variable can be written in the form i.e. interval, probability function or membership function. To perform calculations on the uncertain data the interval arithmetic has to be used. Uncertain variables are used to model the reality in many fields of science i.e. economics, medicine, technology and so on.

The basic arithmetic for uncertainty theory [1] is interval arithmetic. The interval arithmetic is used in grey systems [4], granular computing [8], fuzzy systems [2, 7]. There exist many versions of interval arithmetic but the most known and often used in practice is Moore interval arithmetic [5, 6]. Moore interval arithmetic as a solution gives an interval so it is one dimensional arithmetic. In many problems the solution described in one dimensional form is just a part of full solution. The alternative for one dimensional arithmetic is Relative Distance Measure (RDM) multidimensional interval arithmetic. The RDM multidimensional interval arithmetic was elaborated by A. Piegat and is developed in collaboration with M. Landowski [9, 10, 11, 12, 3].

The article presents the RDM interval arithmetic in the comparison with the results obtained by Moore interval arithmetic and global optimization (Interval Solver 2000). Examples and results of Moore arithmetic and global optimization were taken from [13]. On the basis of the given problems the solution by RDM multidimensional interval arithmetic were obtained.

RDM multidimensional interval arithmetic

In the RDM interval arithmetic the given element $x$ of interval $X = [x, \bar{x}]$, where $x \leq \bar{x}$, is described using the RDM variable $\alpha_x$, where $\alpha_x \in [0; 1]$, formula (1).

$$x = x + \alpha_x (\bar{x} - x)$$

(1)

The interval $X = [x, \bar{x}]$ in the RDM notation is described as (2).

$$X = \{x: x = x + \alpha_x (\bar{x} - x), \alpha_x \in [0; 1]\}$$

(2)

For example the interval $A = [3; 5]$ in the RDM notation takes form of (3).

$$A = \{a: a = 3 + 2\alpha_a, \alpha_a \in [0; 1]\}$$

(3)

The interval $A = [3; 5]$ is a set of elements $a \in A$ in the form (4).

$$a = 3 + 2\alpha_a, \alpha_a \in [0; 1]$$

(4)

For $\alpha_a = 0$ element $a = 3$ is the lower limit of interval $A$, for $\alpha_a = 1$ element $a = 5$ is the upper limit of interval $A$.

For intervals $X = [\underline{x}, \bar{x}] = \{x: x = \underline{x} + \alpha_x (\bar{x} - \underline{x}), \alpha_x \in [0; 1]\}$ and $Y = [\underline{y}, \bar{y}] = \{y: y = \underline{y} + \alpha_y (\bar{y} - \underline{y}), \alpha_y \in [0; 1]\}$, the basic operation in RDM interval arithmetic on intervals are defined (5) – (8).

Addition

$$X + Y = \{x + y: x + y = [x + \alpha_x (\bar{x} - x) + \underline{y} + \alpha_y (\bar{y} - \underline{y})], \alpha_x, \alpha_y \in [0; 1]\}$$

(5)

Subtraction

$$X - Y = \{x - y: x - y = [x + \alpha_x (\bar{x} - x)] - [\underline{y} + \alpha_y (\bar{y} - \underline{y})], \alpha_x, \alpha_y \in [0; 1]\}$$

(6)

Multiplication

$$X \cdot Y = \{x \cdot y: x \cdot y = [x + \alpha_x (\bar{x} - x)] \cdot [\underline{y} + \alpha_y (\bar{y} - \underline{y})], \alpha_x, \alpha_y \in [0; 1]\}$$

(7)

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2. Reviewed paper.
Division

\[ X/Y = \left\{ x/y : x/y = \frac{x + \alpha_x(x - \bar{x})}{y + \alpha_y(y - \bar{y})} \right\} \text{ if } 0 \notin Y \]  \hspace{1cm} (8)

In RDM multidimensional interval arithmetic it is possible to find the span of the solution even if the results of calculation

In the RDM multidimensional interval arithmetic it is possible to find the span of the solution even if the results of calculation
is in multidimensional space. For interval \( X = [\bar{x}, \bar{x}] \) and \( Y = [\bar{y}, \bar{y}] \) and for the basic operation \(* \in \{+,-,/,\} \) the span is an interval and is defined as (9):

\[ s(X * Y) = [\min(X * Y) ; \max(X * Y)] \], \hspace{1cm} (9)

the operation / is defined only if \( 0 \notin Y \).

**Problem with independent variables**

The problem is called “a simple case from managerial economics” [13]. Example consider the company that is selling the

The problem is called “a simple case from managerial economics” [13]. Example consider the company that is selling the
product at price \( p \) and quantity \( q \). In the first step we assume that the price and quantity are independent variables. The turnover

\[ TR = TR(p, q) = pq, \] \hspace{1cm} (10)

the variable cost \( VC \) is given by (11):

\[ VC = VC(q) = 20q, \] \hspace{1cm} (11)

and the profit \( \pi \) is described by (12):

\[ \pi = \pi(p, q) = TR(p, q) - VC(q) = pq - 20q. \] \hspace{1cm} (12)

In the example variables price and quantity are uncertain and independent variable, the uncertainty is described by interval. Intervals for uncertain variables \( p \) and \( q \) are presented in (13):

\[ p = [55; 65], q = [350; 450]. \] \hspace{1cm} (13)

The solution obtained by Moore interval arithmetic and global optimization depends on the method of calculation. The three cases are considered [13]:

- The first case \( \pi = TR - VC \). The turnover \( TR \) and variable cost \( VC \) by Moore interval arithmetic are calculated using formulas (10) and (11) and then the uncertain profit \( \pi \) is obtained as a subtraction \( TR \) and \( VC \).

- The second case \( \pi = pq - 20q \). The uncertain profit \( \pi \) is calculated using formula (12).

- The third case \( \pi = q(p - 20) \). The uncertain profit \( \pi \) is calculated using (12) in the form of (14):

\[ \pi = \pi(p, q) = q(p - 20). \] \hspace{1cm} (14)


**Tab. 1. Results of the profit \( \pi \) obtained by Moore interval arithmetic and global optimization for three different method of calculation, the case with independent variables**

<table>
<thead>
<tr>
<th>Method of calculation</th>
<th>The first case ( \pi = TR - VC )</th>
<th>The second case ( \pi = pq - 20q )</th>
<th>The third case ( \pi = q(p - 20) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moore IA</td>
<td>[10250; 22250]</td>
<td>[10250; 22250]</td>
<td>[12250; 20250]</td>
</tr>
<tr>
<td>Global optimization</td>
<td>[10250; 22250]</td>
<td>[12250; 20250]</td>
<td>[12250; 20250]</td>
</tr>
</tbody>
</table>

Source: [13].

To find the solution obtained by RDM interval arithmetic first the interval \( p \) and \( q \) should be written using RDM variable \( \alpha_p \) and \( \alpha_q \). In RDM notation uncertain variables are presented by formula (15) and (16).

\[ p = \{ p : p = 55 + 10\alpha_p, \alpha_p \in [0; 1] \} \] \hspace{1cm} (15)

\[ q = \{ q : q = 350 + 100\alpha_q, \alpha_q \in [0; 1] \} \] \hspace{1cm} (16)

The first case \( \pi = TR - VC \)

In the RDM multidimensional interval arithmetic the turnover \( TR \) is calculated using (15) and (16), the result is given by (17).
\[ TR = pq = \{ tr: tr = pq = (55 + 10\alpha_p)(350 + 100\alpha_q), \alpha_p, \alpha_q \in [0; 1]\}\]
\[ = \{ tr: tr = 19250 + 3500\alpha_p + 5500\alpha_q + 1000\alpha_p\alpha_q, \alpha_p, \alpha_q \in [0; 1]\}\]  

The variable cost VC using RDM interval arithmetic is calculated as (18).
\[ VC = 20q = \{ vc: vc = 20q = 20(350 + 100\alpha_q), \alpha_q \in [0; 1]\}\]
\[ = \{ vc: vc = 20q = 7000 + 2000\alpha_q, \alpha_q \in [0; 1]\}\]

The variable cost VC obtained by RDM interval arithmetic is one dimensional with one RDM variable \( \alpha_q \), so VC can be presented as an interval (19).
\[ VC=\{ vc: vc = 20q = 7000 + 2000\alpha_q, \alpha_q \in [0; 1]\} = [7000; 9000] \] (19)

Now the profit \( \pi \) can be calculated using (17) and (18). The profit is calculated in (20).
\[ \pi = TR - VC \]
\[ = \{ \pi: \pi = 19250 + 3500\alpha_p + 5500\alpha_q + 1000\alpha_p\alpha_q - 7000 + 2000\alpha_q, \alpha_p, \alpha_q \in [0; 1]\}\]
\[ = \{ \pi: \pi = 12250 + 3500\alpha_p + 3500\alpha_q + 1000\alpha_p\alpha_q, \alpha_p, \alpha_q \in [0; 1]\}\]  

To find the graphical representation of solution \( \pi \) and variable TR extreme and border values should be calculated. For monotonic functions \( \pi \) and TR border values give the graphical representation of solution, table 2.

<table>
<thead>
<tr>
<th>( \alpha_p )</th>
<th>( p )</th>
<th>0</th>
<th>55</th>
<th>0</th>
<th>55</th>
<th>1</th>
<th>65</th>
<th>1</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_q )</td>
<td>0</td>
<td>1</td>
<td></td>
<td>0</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( q )</td>
<td>350</td>
<td>450</td>
<td></td>
<td>350</td>
<td></td>
<td>450</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TR</td>
<td>1925</td>
<td>2475</td>
<td></td>
<td>2275</td>
<td></td>
<td>2925</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Profit ( \pi )</td>
<td>12250</td>
<td>15750</td>
<td></td>
<td>15750</td>
<td></td>
<td>20250</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The figure 1 shows the graphical representation of result (17) for uncertain variables TR.

![Graphical representation of TR](image)

**Fig. 1. The graphical result obtained by RDM arithmetic for uncertain variable \( TR = pq \)**

In the figure 2 the graphical form of solution (20) obtained by RDM multidimensional interval arithmetic is presented.
The span of the profit $\pi$ obtained by RDM interval arithmetic is presented in (21).

$$s(\pi) = \left[ \begin{array}{c}
\min_{\alpha_p, \alpha_q \in [0; 1]} (12250 + 3500\alpha_p + 3500\alpha_q + 1000\alpha_p\alpha_q) \\
\max_{\alpha_p, \alpha_q \in [0; 1]} (12250 + 3500\alpha_p + 3500\alpha_q + 1000\alpha_p\alpha_q)
\end{array} \right]$$

$$s(\pi) = s(TR - VC) = [12250; 20250]$$  \hspace{1cm} (21)

The second case $\pi = pq - 20q$

Using uncertain variable in the notation RDM formula (15) and (16), the solution in the second case is given in (22).

$$\pi = pq - 20q = \{\pi: \pi = (55 + 10\alpha_p)(350 + 100\alpha_q) - 20(350 + 100\alpha_q), \alpha_p, \alpha_q \in [0; 1]\}$$

$$\pi = 19250 + 3500\alpha_q + 5500\alpha_q + 1000\alpha_p\alpha_q - 7000 - 2000\alpha_q, \alpha_p, \alpha_q \in [0; 1]\}$$

$$\pi = \{\pi: \pi = 12250 + 3500\alpha_p + 3500\alpha_q + 1000\alpha_p\alpha_q, \alpha_p, \alpha_q \in [0; 1]\}$$  \hspace{1cm} (22)

The solution obtained in the second case (22) is equal to the solution calculated in the first case (20).

The third case $\pi = q(p - 20)$

For calculation the solution in the third case we use variable described in RDM notation (15) and (16). The solution is presented in formula (23)

$$\pi = q(p - 20) = \{\pi: \pi = (350 + 100\alpha_q)(55 + 10\alpha_p - 20), \alpha_p, \alpha_q \in [0; 1]\}$$

$$\pi = 19250 + 3500\alpha_q + 5500\alpha_q + 1000\alpha_p\alpha_q - 7000 - 2000\alpha_q, \alpha_p, \alpha_q \in [0; 1]\}$$

$$\pi = \{\pi: \pi = 12250 + 3500\alpha_p + 3500\alpha_q + 1000\alpha_p\alpha_q, \alpha_p, \alpha_q \in [0; 1]\}$$  \hspace{1cm} (23)

As we see in (23) the result is equal to the solutions obtained in the first and the second case formulas (20) and (22).

The calculations shows that in RDM interval arithmetic the results are equal in every three cases, the solution does not depends on the form of the problem. The results obtained from Moore and global optimization shows that the solution can be different and dependent on the formulation of the same problem, table 1.

The solution obtained by RDM interval arithmetic is multidimensional, see fig. 2, but the result from Moore arithmetic and global optimization are one dimensional expressed in the form of interval. The interval in some more complicated problems is only the span of solution. Therefore one dimensional methods of interval arithmetic can find only the part of the solution.
Problem with interdependent variables

Similarly as in the problem with independent variables the company is selling one product at price $p$ and quantity $q$ [13]. In this example the assumption is that the price $p$ and the quantity $q$ are interdependent variables. Now variables are described by the demand function (24):

$$p(q) = 100 - 0.1q \quad \text{and} \quad q(p) = 1000 - 10p,$$

(24)

the turnover $TR$ is defined as (25):

$$TR = TR(p) = -10p^2 + 1000p,$$

(25)

the variable cost $VC$ is presented by formula (26):

$$VC = VC(p) = -200p + 20000,$$

(26)

and the profit $\pi$ in the case where variables are interdependent is described as (27):

$$\pi = \pi(p) = TR(p) - VC(p) = -10p^2 + 1200p - 20000.$$

(27)

The price $p$ is an uncertain variable described by interval (28):

$$p = [55; 65].$$

(28)

The solution will be calculated in two forms [13]:

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The first case $\pi = TR - VC$. The turnover $TR$ and variable cost $VC$ by Moore interval arithmetic are calculated using formulas (25) and (26) and then the uncertain profit $\pi$ is obtained as a subtraction $TR$ and $VC$.

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The second case $\pi = -10p^2 + 1200p - 20000$. The uncertain profit $\pi$ is calculated using formula (27).


<table>
<thead>
<tr>
<th>Method of calculation</th>
<th>The first case $\pi = TR - VC$</th>
<th>The second case $\pi = -10p^2 + 1200p - 20000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moore IA</td>
<td>$[3750; 27750]$</td>
<td>$[3750; 27750]$</td>
</tr>
<tr>
<td>Global optimization</td>
<td>$[13750; 17750]$</td>
<td>$[15750; 16000]$</td>
</tr>
</tbody>
</table>

Source: [13].

To find the solution by RDM interval arithmetic first the uncertain variable $p$ should be described using RDM variable $\alpha_p$ ($\alpha_p \in [0; 1]$). The interval (28) in the RDM notation take the form of (29).

$$p = \{p: p = 55 + 10\alpha_p, \alpha_p \in [0; 1]\}$$

(29)

The first case $\pi = TR - VC$

By RDM interval arithmetic the turnover $TR$ is calculated using formula (25) and takes the form of (30).

$$TR = \{tr: tr = -10(\alpha_p + 55)^2 + 1000(\alpha_p + 55), \alpha_p \in [0; 1]\}$$

$$= \{tr: tr = -10(3025 + 1100\alpha_p + 1000\alpha_p^2) + 55000 + 10000\alpha_p, \alpha_p \in [0; 1]\}$$

$$= \{tr: tr = -30250 - 11000\alpha_p - 10000\alpha_p^2 + 55000 + 10000\alpha_p, \alpha_p \in [0; 1]\}$$

$$= \{tr: tr = 24750 - 1000\alpha_p - 1000\alpha_p^2, \alpha_p \in [0; 1]\}$$

(30)

The variable cost $VC$ is calculated by (26), the result presents (31).

$$VC = \{tr: tr = -200(\alpha_p + 55) + 20000, \alpha_p \in [0; 1]\}$$

$$= \{tr: tr = 9000 - 2000\alpha_p, \alpha_p \in [0; 1]\}$$

(31)

The solution obtained by RDM interval arithmetic for case with interdependent variable is calculated in (32).

$$\pi = TR - VC = \{\pi: \pi = 24750 - 1000\alpha_p - 1000\alpha_p^2 - 9000 + 2000\alpha_p, \alpha_p \in [0; 1]\}$$

$$= \{\pi: \pi = 15750 + 1000\alpha_p - 1000\alpha_p^2, \alpha_p \in [0; 1]\}.$$

(32)
Obtained solution profit $\pi$ is one dimensional with RDM variable $\alpha_p \ (\alpha_p \in [0; 1])$, so the result will be described by interval (34) using formula (33).

$$\pi = \left[ \min_{\alpha_p \in [0; 1]} \pi ; \ \max_{\alpha_p \in [0; 1]} \pi \right]$$

$$\pi = [15750; 16000]$$  \hspace{1cm} (33)  \hspace{1cm} (34)

The second case $\pi = -10p^2 + 1200p - 20000$

The solution of the problem with interdependent variables $p$ and $q$ is calculated using formula (27), where uncertain interval $p$ is written in RDM notation (29). The form (35) presents profit $\pi$ calculated by RDM interval arithmetic.

$$\pi = -10p^2 + 1200p - 20000$$

$$= \{\pi : \pi = -10(55 + 10\alpha_p)^2 + 1200(55 + 10\alpha_p) - 20000, \ \alpha_p \in [0; 1]\}$$

$$= \{\pi : \pi = -10(3025 + 1100\alpha_p + 100\alpha_p^2) + 66000 + 1200\alpha_p - 20000, \ \alpha_p \in [0; 1]\}$$

$$= \{\pi : \pi = 15750 + 1000\alpha_p - 1000\alpha_p^2, \ \alpha_p \in [0; 1]\}$$  \hspace{1cm} (35)

The interval obtained in (35) is equal to solution in the first case (32), so the solution in this method of calculation is equal to the interval (34).

As in the case with independent variables problem now we again obtained that the solution from RDM interval arithmetic is equal in the two different formulation of problem. The solution obtained by RDM arithmetic is equal to the solution of global optimization in the second case. The solution of Moore interval arithmetic has a large span it is because of many operation made on intervals. In the Moore interval arithmetic successive operations on intervals increase the span of the solution.

Conclusions

The article presents the RDM multidimensional interval arithmetic and the usage of this arithmetic in the economic problem. The obtained solution were compared with the solution from Moore interval arithmetic and global optimization results. The results show that the solution in Moore arithmetic and global optimization depend on the formulation of the problem. In different forms of a problem the Moore arithmetic and global optimization method give different results. In RDM interval arithmetic different forms of the problem give the same solution. It suggests that Moore arithmetic and global optimization cannot correctly and credibly solve more complicated problems. The results of Moore interval arithmetic and global optimization are one dimensional and give only span of the solution, except one-dimensional problems where the solution is an interval. The RDM interval arithmetic is multidimensional and gives the full solution of the problem.

Abstract

Uncertain variables are often used for solving realistic problems. To find the solution of a realistic problem the model with uncertain variable has to be built. Based on the model with uncertain variables operations on intervals are necessary. The article presents the multidimensional RDM interval arithmetic and its application to solving an economic problem. Obtained solutions are compared with results from the one dimensional Moore interval arithmetic and global optimization.

REFERENCES


